Circular polarization induced by scintillation in a magnetized medium

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A theory is presented for the development of circular polarization as radio waves propagate through the turbulent, birefringent interstellar medium. The fourth order moments of the wave field are calculated, and it is shown that unpolarized incident radiation develops a nonzero variance in circular polarization. A magnetized turbulent medium causes the Stokes parameters to scintillate in a nonidentical manner. A specific model for this effect is developed for the case of density fluctuations in a uniform magnetic field.

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I. INTRODUCTION

Circular polarization observed in radio emission from pulsars and quasars is not understood. Although the emission mechanisms for these two classes of sources are quite different, both have the common feature that the emission is due to highly relativistic particles in a magnetic field, for which the polarization should be predominantly linear with a circular component of order the inverse of the Lorentz factor of the radiating particles. As discussed further below, this intrinsic component of circular polarization does not account for the observations. We propose that the circular polarization is imposed as a propagation effect, due the scintillations in the interstellar medium (ISM) [1] which account well for many of the observed, and otherwise unexplained, time variations in pulsars and quasars [2,3]. The underlying physics for this alternative explanation is presented in this paper. We propose to discuss the details of the astrophysical applications elsewhere.

Scintillations are attributed to scattering of the radio waves off density inhomogeneities associated with turbulence in the ISM. The data indicate a power-law model for the turbulence, with the power-law index consistent with the Kolmogorov value ($\beta=11/3$ in the notation used here) [4]. The dataset on pulsars is sufficiently large to allow mapping of the turbulence across the galaxy [5], implying much stronger scattering at low galactic latitudes, where most pulsars are observed, than at high galactic latitudes, where most quasars are observed. A planar wave front becomes rippled as it traverses the region of turbulence. Two length scales play a central role in the theory: the Fresnel scale $r_F=(\lambda D/2\pi)^{1/2}$, where λ is the wavelength of the radio wave and D is the distance between the observer and the scattering region; and the diffractive scale $r_{\rm diff}$, over which the fluctuations in the phase decorrelate due to the turbulence. Physically, $r_{\rm diff}$ characterizes the sizes of the ripples, and r_F characterizes the size of the coherent patch an observer can see on the unrippled wave front when only the geometric phase difference is taken into account. ''Weak scattering'' corresponds to $r_F \ll r_{\rm diff}$, when an observer sees a single coherent patch that is slightly tilted (image displacement) and slightly convex (focused) or concave (defocused). ''Strong scattering'' corresponds to $r_F \gg r_{\rm diff}$, when an observer sees many coherent patches of size $r_{\rm diff}$ within an envelope of size $r_{\rm ref}=r_F^2/r_{\rm diff}$, and multipath propagation occurs [6]. Intensity variations in strong scattering are induced by both diffractive effects, due to interference between the coherent patches, and refractive effects, caused by focusing (defocusing) of the ray bundle due to phase curvature across the scattering disk. There is a transition between strong scattering at lower frequencies and weak scattering at higher frequencies. For most pulsars the transition frequency is higher than the frequencies for which data are available, and for quasars the observations span the expected transition frequency, ~7 GHz [7].

Inclusion of the interstellar magnetic field implies that the ISM is birefringent, and propagation of radiation depends upon its polarization. In a homogeneous birefringent medium, radiation separates into the two oppositely polarized natural wave modes, with the phase difference between them increasing linearly with propagation distance. The two wave fronts corresponding to the two modes become systematically displaced from each other with increasing distance. The natural modes in the ISM are circularly polarized to an excellent approximation, and the birefringence results in Faraday rotation of the plane of linear polarization of any incident radiation. The amount of Faraday rotation is parametrized in terms of the rotation measure (RM), which is defined such that the phase difference between the two modes is $\mathrm{RM}\lambda^2$ [8]. Inhomogeneity combined with birefringence implies that the wave fronts are both rippled and displaced from each other. One implication is that when the wave fronts are recombined, there is a random component in the phase difference, and the resulting stochastic Faraday rotation is characterized by a variance in RM [9].

The main point made in the present paper is that scattering in the magnetized ISM necessarily leads to a component of circular polarization (CP) in the observed radiation. This arises because any lateral displacement of the wave front implies that the ripples are not superimposed when the two wave fronts are combined. As a result, there are alternative patches of excess right hand (RH) and excess left hand (LH) circular polarization on the image in the observer's plane. In this paper we present a detailed theory for such scintillation-induced CP. We argue that the predicted features of the CP are sufficiently promising to warrant the development of de-

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tailed models for the observed CP in pulsars and quasars based on it.

CP is observed in both pulsars [10] and in some compact extragalactic sources [11–14]. Most of the data on pulsars are integrated over many pulses, and the integrated pulse profile typically shows relatively small CP. However, there is evidence that in at least a few pulsars for which individual pulses can be studied, the CP is relatively large in individual pulses and varies from pulse to pulse such that the integrated value is much smaller than the typical value. There is no satisfactory explanation for the CP [15]. A small ($\lesssim 0.1\%$ to a few percent) but significant degree of CP is observed in some compact extragalactic sources [11–14]. The suggested interpretations include the intrinsic polarization associated with synchrotron radiation [16], and partial conversion of linear into circular polarization due to the ellipticity of the natural wave modes of the cold background plasma [17] or of the relativistic electron gas itself [18–20]. However none of these suggested interpretations has proved satisfactory in accounting for (a) the frequency dependence, (b) the temporal variations, and (c) the magnitude of the observed circular polarization [21]. The explanation proposed in this paper is as outlined above. Specifically, for a source with zero intrinsic CP seen through a turbulent magnetized plasma, there is a variable CP which corresponds to a zero average of the Stokes parameter V, $\langle V\rangle=0$, but a nonzero variance, $\langle V^2\rangle \neq 0$. Our initial objective is to use the theory of scintillations in a magnetized plasma [22–25] to calculate $\langle V^2\rangle$.

The magnitude of the expected value of the CP needs to be of the same order of magnitude as the observed CP for the theory proposed here to be relevant. For pulsars, the CP can be several tens of percent, but it may be that some of this CP results from birefringence in the pulsar magnetosphere itself, which we do not consider in detail here. For the most extreme case for quasars, the observed CP can be several percent, which is relatively high because there is independent evidence suggesting that the varying (scintillating in our interpretation) part of the source is only a modest fraction of the entire source. Hence the observations, in the most extreme cases, suggest that the CP of the scintillating component can be as high as a few tens of percent. In the theory developed here, most of the terms that contribute to the CP are very small, of order the ratio of the cyclotron frequency in the ISM to the radio frequency (typically $\sim 10^{-8}$), and can be of no practical interest. However, the effect on which we concentrate can give arbitrarily high CP. This effect is due to birefringent refraction causing an angular separation between the emerging rays in the LH and RH polarized wave modes. The displacement of the centroids of the LH and RH images increases linearly with the distance from the screen where the birefringent refraction occurs. It is possible for the LH and RH images not to overlap, resulting in patches of 100% CP. The angular deviation required to produce relatively large CP is determined by the ratio of the characteristic size of the scintillation pattern divided by the distance between the observer and the screen where the birefringent refraction occurs. Although the angular separation between the rays is always extremely small, we argue elsewhere [26] that observed gradients in RM imply birefringent refraction through a sufficiently large angle to satisfy the criterion that observable CP be produced.

Our specific assumptions are explained in Sec. 2, where the wave equation is reduced to the form used in treating the scattering. The mutual coherence of a polarized wavefield is derived in Sec. 3, and is shown to reproduce some known results [27,28]. In Sec. 4 the second-order correlations of the Stokes parameters are derived from the fourth-order moment of the wave field, and explicit solutions are obtained in the thin-screen approximation. In Sec. 5 we discuss scintillation-induced CP. The conclusions are presented in Sec. 6.

II. PROPAGATION THROUGH A MAGNETIZED PLASMA

In this section the propagation of radiation through a magnetized stochastic medium is related to its effect upon the electric field of the wave. We start with propagation through a homogeneous weakly anisotropic medium and then generalize to include the effect of inhomogeneities in the scattering medium. For a weakly inhomogeneous medium the two wave modes are assumed to be transverse to a zeroth approximation (the isotropic limit) and the degeneracy between the two transverse states of polarization is broken by the weak anisotropy, to a first approximation. In the first approximation the two modes have slightly different refractive indices, and this approximation suffices to treat Faraday rotation and all the effects of interest here.

The wave equation projected onto the transverse plane is [24]

$$\left(-k^2\delta^{\alpha\beta}+\frac{\omega^2}{c^2}K^{\alpha\beta}(\omega,\mathbf{k})\right)A^{\beta}(\omega,\mathbf{k})=0, \tag{2.1}$$

where $A^{\alpha}(\omega,\mathbf{k})$ is the wave amplitude, $K^{\alpha\beta}(\omega,\mathbf{k})$ is the dielectric tensor, and the greek indices run over the two transverse coordinates. We write

$$K^{\alpha\beta}=\langle K^{\alpha\beta}\rangle+\delta K^{\alpha\beta}, \tag{2.2}$$

where the angular brackets denote the mean and $\delta K^{\alpha\beta}$ denotes a fluctuating part with a mean of zero, $\langle\delta K^{\alpha\beta}\rangle=0$. The transverse components of the dielectric tensor may be expressed in terms of the Pauli matrices, and for the average part we write

$$\langle K^{\alpha\beta}\rangle=K_A\sigma_A^{\alpha\beta}, \tag{2.3}$$

where the sum over $A=[I,Q,U,V]$ is implied, with

$$\sigma_I^{\alpha\beta}=\delta^{\alpha\beta}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \sigma_Q^{\alpha\beta}=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$
$$\sigma_U^{\alpha\beta}=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_V^{\alpha\beta}=\begin{pmatrix}0&-i\\i&0\end{pmatrix}. \tag{2.4}$$

In the discussion here we neglect any dissipation, which implies that we retain only the Hermitian part of $K^{\alpha\beta}$, so the K_A are all real.

The two modes are labeled $\sigma=\pm$. The mode σ has a wave number $k_\sigma=n_\sigma\omega/c$, where n_σ is the refractive index for waves in mode σ, and polarization vector $\mathbf{e}_\sigma$. It is straightforward to solve for the dispersion relations, which are

$$k_\sigma^2 = \frac{\omega^2}{c^2}\{K_I + \sigma[K_Q^2 + K_U^2 + K_V^2]^{1/2}\}. \tag{2.5}$$

The quantity K_I is the dielectric constant in the isotropic approximation, with $K_Q^2 + K_U^2 + K_V^2 \ll K_I^2$ in the weak anisotropy limit. The polarization vectors in the two-dimensional transverse plane are

$$\mathbf{e}_\sigma = \frac{(K_Q + \sigma[K_Q^2 + K_U^2 + K_V^2]^{1/2}, K_U + iK_V)}{\{2[K_Q^2 + K_U^2 + K_V^2]^{1/2}(K_Q + \sigma[K_Q^2 + K_U^2 + K_V^2]^{1/2})\}^{1/2}}. \tag{2.6}$$

One is always free to orient the coordinate axes such that $K_U = 0$. On doing so, Eq. (2.6) may be written in the form

$$\mathbf{e}_\sigma = (1, iT_\sigma), \quad T_\sigma = \frac{K_V}{K_Q + \sigma[K_Q^2 + K_V^2]^{1/2}}, \quad T_+ T_- = -1, \tag{2.7}$$

where T_σ is the axial ratio of the polarization ellipse in the mode σ. In the case of Faraday rotation, the wave modes are circularly polarized, which corresponds to $|T_\sigma| = 1$. The circular polarizations are

$$\mathbf{e}_{r,l} = \frac{1}{\sqrt{2}}(1, \pm i), \tag{2.8}$$

where r and l refer to right and left hands, respectively. In this paper we consider only the circularly polarized approximation explicitly, but the general theory is valid for any T_σ.

In scattering theory it is conventional to make the parabolic approximation to the wave equation [29,30]. This approximation is related to the paraxial approximation in geometric optics, in the sense that there is a favored ray direction (the z axis in our case), and that only small deviations from it are considered. In the parabolic approximation the wave field is written as the product of a fast varying term $e^{ik_\sigma z}$ and a term $u^\alpha(z,\mathbf{r})$ that is assumed to be a slowly varying function of z in the sense that its second derivative with respect to z may be neglected. The difference between k_+ and k_- is introduced explicitly by writing $k_\sigma = k + \sigma\delta k$, which defines δk. In a homogeneous medium the two phase factors $ik_\sigma z$ include the mean phase ikz, corresponding to the average over the two modes, and the phase difference $\pm i\delta kz$ between the components in the two modes relative to this mean.

The inhomogeneities are introduced through a fluctuating part $\delta K^{\alpha\beta}$ of the dielectric tensor. On making the parabolic approximation to Eq. (2.1), one obtains the following equations for the propagation of the right- and left-hand polarized wave fields

$$\begin{aligned}&\left(2ik\frac{\partial}{\partial z} + \nabla_\perp^2 + \frac{\omega^2}{c^2}\delta K_+^2\right)\tilde{u}_+ + \frac{\omega^2}{c^2}\delta K_{+-}\tilde{u}_- e^{-2i\delta kz} = 0,\\ &\left(2ik\frac{\partial}{\partial z} + \nabla_\perp^2 + \frac{\omega^2}{c^2}\delta K_-^2\right)\tilde{u}_- + \frac{\omega^2}{c^2}\delta K_{-+}\tilde{u}_+ e^{2i\delta kz} = 0,\end{aligned} \tag{2.9}$$

where the perturbation terms involve

$$\delta K_{\sigma\sigma'} = e_\sigma^{*\alpha} e_{\sigma'}^{\beta}\delta K^{\alpha\beta}, \tag{2.10}$$

and

$$A^\alpha = \sum_{\sigma=\pm} e^{ikz}\mathbf{e}_\sigma \tilde{\mathbf{u}}_\sigma(z,\mathbf{r}), \quad \tilde{\mathbf{u}}_\sigma(z,\mathbf{r}) = \mathbf{e}_\sigma u_\sigma(z,\mathbf{r}) e^{i\sigma\delta kz}. \tag{2.11}$$

The terms involving δK_{+-} and δK_{-+} are zero if the inhomogeneities do not affect the polarization of the natural modes, that is if $\delta K^{\alpha\beta} - \sigma_I^{\alpha\beta}\delta K_I$ is proportional to $\langle K^{\alpha\beta}\rangle - \sigma_I^{\alpha\beta}\langle K_I\rangle$. This is the case to an excellent approximation if the fluctuations do not involve the direction of the magnetic field, and even for fluctuations that affect the direction of the magnetic field it is an excellent approximation provided that the modes are nearly circularly polarized. The extreme conditions under which these terms might be non-negligible are ignored here.

Neglecting the terms involving δK_{+-} and δK_{-+}, the following relations describe the propagation of the wave amplitude through a weakly anisotropic inhomogeneous medium:

$$\begin{aligned}&\left(2ik\frac{\partial}{\partial z} + \nabla_\perp^2 + \frac{\omega^2}{c^2}\delta K^2{}_+\right)\tilde{u}_+ = 0,\\ &\left(2ik\frac{\partial}{\partial z} + \nabla_\perp^2 + \frac{\omega^2}{c^2}\delta K^2{}_-\right)\tilde{u}_- = 0.\end{aligned} \tag{2.12}$$

Equations (2.12) are used as the basis for the theory developed below.

III. SECOND-ORDER MOMENTS OF THE WAVE FIELD

In this section we calculate the average visibilities in each of the four Stokes parameters, thus determining the properties of the average image of a scattered source.

A. Ensemble averaged visibilities

Consider a two-element interferometer with receivers located at positions $\mathbf{r}_1$ and $\mathbf{r}_2$, that measures the left- and right-hand circularly polarized components of the electric field. We define the visibility in a given polarization using the generalized second-order moment of the electric field:

$$\gamma_{\sigma\sigma'}(z;\mathbf{r}_1,\mathbf{r}_2) = \langle u_\sigma(z;\mathbf{r}_1) u_{\sigma'}^*(z;\mathbf{r}_2)\rangle. \tag{3.1}$$

The Stokes parameters are defined in terms of the left- and right-hand circularly polarized components of the electric field for $\mathbf{r}_1 = \mathbf{r}_2$: $I^2{}_+ = u_+ u_+^*$, $I^2{}_- = u_- u_-^*$, $I_{+-} = u_+ u_-^*$, and $I_{-+} = u_- u_+^*$. These are related to the conventional Stokes parameters I, Q, U, and V by

$$\begin{aligned}&I = \frac{1}{2}(I^2{}_+ + I^2{}_-), \quad Q = \frac{1}{2}(I_{+-} + I_{-+}),\\ &U = i\frac{1}{2}(I_{+-} - I_{-+}), \quad V = \frac{1}{2}(I^2{}_+ - I^2{}_-),\end{aligned} \tag{3.2}$$

$$I^2{}_+ = I + V, \quad I^2{}_- = I - V, \quad I_{+-} = Q - iU, \quad I_{-+} = Q + iU.$$

In this notation the Stokes visibilities are

$$I(z;\mathbf{r}_1,\mathbf{r}_2)=\frac{1}{2}[\gamma^2{}_+(z;\mathbf{r}_1,\mathbf{r}_2)+\gamma^2{}_-(z;\mathbf{r}_1,\mathbf{r}_2)],$$

$$Q(z;\mathbf{r}_1,\mathbf{r}_2)=\frac{1}{2}[\gamma_{+-}(z;\mathbf{r}_1,\mathbf{r}_2)+\gamma_{-+}(z;\mathbf{r}_1,\mathbf{r}_2)],$$

$$U(z;\mathbf{r}_1,\mathbf{r}_2)=i\frac{1}{2}[\gamma_{+-}(z;\mathbf{r}_1,\mathbf{r}_2)-\gamma_{-+}(z;\mathbf{r}_1,\mathbf{r}_2)], \tag{3.3}$$

$$V(z;\mathbf{r}_1,\mathbf{r}_2)=\frac{1}{2}[\gamma^2{}_+(z;\mathbf{r}_1,\mathbf{r}_2)-\gamma^2{}_-(z;\mathbf{r}_1,\mathbf{r}_2)].$$

Using Eqs. (2.12) and their complex conjugates, the four visibilities obey the four propagation equations obtained by setting $\sigma=\pm 1$ and $\sigma'=\pm 1$ in

$$\left[2ik\frac{\partial}{\partial z}+\nabla_1^2-\nabla_2^2+k^2\delta K_{\sigma\sigma}(\mathbf{r}_1)-k^2\delta K_{\sigma'\sigma'}(\mathbf{r}_2)\right]\times\gamma_{\sigma\sigma'}(z;\mathbf{r}_1,\mathbf{r}_2)e^{i\delta kz(\sigma'-\sigma)}=0. \tag{3.4}$$

Due to both the properties of the radiation from the source and the stochastic nature of the phase screen, it is desirable to compute the ensemble average visibility, denoted by $\Gamma_{\sigma\sigma'}$. The ensemble average is considered an average over both time and over the phase fluctuations. It is normally assumed that the phase screen itself is static, with any perceived time variability due to relative motion between the screen and the source-observer line of sight at some velocity $\mathbf{v}$. This is known as the frozen screen approximation. An observer measures the visibility function at time t to be $\gamma_{\sigma\sigma'}(z;\mathbf{r}_1+\mathbf{v}t,\mathbf{r}_2+\mathbf{v}t)$. However, if the system is homogeneous this visibility depends only on $\mathbf{r}_1+\mathbf{v}t-\mathbf{r}_2-\mathbf{v}t$, and is therefore independent of t. Thus the average over time is trivial and the visibility is a function of relative receiver separation only.

On replacing the independent variables $\mathbf{r}_1$ and $\mathbf{r}_2$ by $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ and $s=1/2(\mathbf{r}_1+\mathbf{r}_2)$, it follows that in a statistically homogeneous medium the average over the fluctuations is a function of $\mathbf{r}$ only. The average over the propagation equation (3.4) then leads to a propagation equation for, $\Gamma_{\sigma\sigma'}(z;\mathbf{r})=\langle\gamma_{\sigma\sigma'}(z;\mathbf{r}+\mathbf{r}',\mathbf{r}')\rangle$:

$$\left[2ik\frac{\partial}{\partial z}+\nabla_{\mathbf{r}\cdot\mathbf{s}}+k^2\xi_{\sigma\sigma'}(\mathbf{r})\right]\Gamma_{\sigma\sigma'}(z;\mathbf{r},\mathbf{s})e^{i\delta kz(\sigma'-\sigma)}=0, \tag{3.5}$$

$$\xi_{\sigma\sigma'}(\mathbf{r},\mathbf{s})=\langle\delta K_{\sigma\sigma}(0)-\delta K_{\sigma'\sigma'}(\mathbf{r})\rangle, \tag{3.6}$$

with $\Delta_{\mathbf{r}\cdot\mathbf{s}}=2(\partial^2/\partial r_x\partial s_x-\partial^2/\partial r_y\partial s_y)$, $\mathbf{r}=r_x\hat{\mathbf{x}}+r_y\hat{\mathbf{y}}$, and $\mathbf{s}=s_x\hat{\mathbf{x}}+s_y\hat{\mathbf{y}}$. Suppose that the phase inhomogeneities are located on a thin screen of thickness Δz. Then to first order in Δz, the visibility measured a distance z from the screen is independent of the screen thickness. It is related to the incident visibility $\Gamma_{\sigma\sigma'}(0;\mathbf{r}')$ according to

$$\Gamma_{\sigma\sigma'}(z;\mathbf{r})=\left(\frac{k}{2\pi z}\right)^2 e^{i\delta k(\sigma'-\sigma)}\int d^2\mathbf{r}'\,d^2\mathbf{s}'\,\Gamma_{\sigma\sigma'}(0;\mathbf{r}')\times\exp\left[\frac{ik(\mathbf{r}-\mathbf{r}')\cdot\mathbf{s}'}{z}+i\Delta\phi_{\sigma\sigma'}(\mathbf{r}')\right], \tag{3.7}$$

$$\Delta\phi_{\sigma\sigma'}(z;\mathbf{r}')=k^2\int_0^z dz'\,\xi_{\sigma\sigma'}(z';\mathbf{r}'). \tag{3.8}$$

B. Average over phase fluctuations

The average over the random fluctuations on the screen is performed under the assumption that the fluctuations in both the isotropic and anisotropic terms are Gaussian. The average $\langle\exp[i\delta x]\rangle=\exp[-\langle(\delta x)^2\rangle/2]$ applies for any Gaussian random variable δx. Writing $\delta\phi_\sigma(\mathbf{r})=\int_0^z dz'(\omega/c)\delta K_{\sigma\sigma}(z';\mathbf{r})$, we make use of this average in defining the generalized phase structure function:

$$D_{\sigma\sigma'}(\mathbf{r})=2[C_{\sigma\sigma'}(0)-C_{\sigma\sigma'}(\mathbf{r})], \tag{3.9}$$

$$C_{\sigma\sigma}(\mathbf{r})=\langle\delta\phi_\sigma(\mathbf{r}')\delta\phi_{\sigma'}(\mathbf{r}'+\mathbf{r})\rangle. \tag{3.10}$$

On specializing to the case where the natural wave modes are circularly polarized, the contribution of the isotropic and anisotropic fluctuations in Eq. (3.9) may be made explicit by introducing the notation $\delta K_{\sigma\sigma}=\delta K_I+\sigma K_V$, where K_I is the isotropic component of the tensor and K_V is the anisotropic component. (The elipticity of the modes is determined by K_Q/K_V, which is set to zero in assuming that the polarizations are circular.) The phase fluctuations may be separated in an identical manner: $\delta\phi_\sigma(\mathbf{r})=\delta\phi_I(\mathbf{r})+\sigma\delta\phi_V(\mathbf{r})$, with ϕ_I and ϕ_V denoting the isotropic and anisotropic components, respectively. Equations (3.9) are expanded as

$$D_\pm^2(\mathbf{r})=D_{II}(\mathbf{r})\pm 2D_{IV}(\mathbf{r})+D_{VV}(\mathbf{r}) \tag{3.11}$$

$$D_{\pm\mp}(\mathbf{r})=D_{II}(\mathbf{r})\mp D_{IV}(\mathbf{r})\pm D_{VI}(\mathbf{r})-D_{VV}(\mathbf{r})+4C_{VV}(0), \tag{3.12}$$

with

$$C_{XY}=\langle\delta\phi_X(\mathbf{r})\delta\phi_Y(\mathbf{r}'+\mathbf{r})\rangle,\quad X,Y=[I,V]. \tag{3.13}$$

The structure function $D_{II}(r)$ represents the effect of the isotropic phase fluctuations. Fluctuations purely in the rotation measure are characterized by D_{VV} which we call the "rotation measure structure function." The terms D_{IV} and D_{VI} represent the cross-correlations between in the isotropic and anisotropic phase fluctuations.

Performing the average over the phase fluctuations and denoting the mean anisotropic phase $\delta kz/2$ introduced in Eq. (2.11) by $-\phi_V$, the ensemble-averaged mutual coherence is

$$\Gamma_{\sigma\sigma'}(z;\mathbf{r})=\left(\frac{k}{2\pi z}\right)^2\int d^2\mathbf{r}'\,d^2\mathbf{s}'\,\Gamma_{\sigma\sigma'}(0;\mathbf{r}')\times\exp\left[\frac{ik(\mathbf{r}-\mathbf{r}')\cdot\mathbf{s}'}{z}-\frac{D_{\sigma\sigma'}(\mathbf{r}')}{2}+i(\sigma-\sigma')\phi_V\right]. \tag{3.14}$$

Equation (3.3) is inverted at $z=0$ to obtain initial values of $\Gamma_{\sigma\sigma'}$ in terms of the Stokes parameters at the screen $z=0$: $I(0;\mathbf{r})$, $Q(0;\mathbf{r})$, $U(0;\mathbf{r})$, and $V(0;\mathbf{r})$. The ensemble-averaged visibilities may therefore be expressed in terms of the initial polarization and the generalized phase structure function as follows:

$$\langle I\rangle(z;\mathbf{r})=\frac{I(0;\mathbf{r})+V(0;\mathbf{r})}{2}\exp\left[-\frac{1}{2}D^2{}_{+}(\mathbf{r})\right]+\frac{I(0;\mathbf{r})-V(0;\mathbf{r})}{2}\exp\left[-\frac{1}{2}D^2{}_{-}(\mathbf{r})\right],$$

$$\langle Q\rangle(z;\mathbf{r})=\frac{Q(0;\mathbf{r})-iU(0;\mathbf{r})}{2}e^{2i\phi_V}\exp\left[-\frac{1}{2}D_{+-}(\mathbf{r})\right]+\frac{Q(0;\mathbf{r})+iU(0;\mathbf{r})}{2}e^{-2i\phi_V}\exp\left[-\frac{1}{2}D_{-+}(\mathbf{r})\right],$$

$$\langle U\rangle(z;\mathbf{r})=\frac{iQ(0;\mathbf{r})+U(0;\mathbf{r})}{2}e^{2i\phi_V}\exp\left[-\frac{1}{2}D_{+-}(\mathbf{r})\right]-\frac{iQ(0;\mathbf{r})-U(0;\mathbf{r})}{2}e^{-2i\phi_V}\exp\left[-\frac{1}{2}D_{-+}(\mathbf{r})\right],$$

$$\langle V\rangle(z;\mathbf{r})=\frac{I(0;\mathbf{r})+V(0;\mathbf{r})}{2}\exp\left[-\frac{1}{2}D^2{}_{+}(r)\right]-\frac{I(0;\mathbf{r})-V(0;\mathbf{r})}{2}\exp\left[-\frac{1}{2}D^2{}_{+}(\mathbf{r})\right]. \tag{3.15}$$

The assumption that the statistics of the phase fluctuations are stationary, implies $\langle\delta\phi_I(\mathbf{r})\delta\phi_V(\mathbf{r}+\mathbf{r}')\rangle=\langle\delta\phi_V(\mathbf{r})\delta\phi_I(\mathbf{r}+\mathbf{r}')\rangle$, and hence $D_{IV}=D_{VI}$. The visibilities reduce to

$$\begin{pmatrix}\langle I\rangle(z;\mathbf{r})\\ \langle Q\rangle(z;\mathbf{r})\\ \langle U\rangle(z;\mathbf{r})\\ \langle V\rangle(z;\mathbf{r})\end{pmatrix}=e^{-D_{II}(\mathbf{r})/2}\begin{pmatrix}e^{-D_{VV}(\mathbf{r})/2}[\langle I\rangle(0;\mathbf{r})\cosh D_{IV}(\mathbf{r})-\langle V\rangle(0;\mathbf{r})\sinh D_{IV}(\mathbf{r})]\\ e^{D_{VV}(\mathbf{r})/2-2C_{VV}(0)}\{\langle Q\rangle(0;\mathbf{r})\cos 2\phi_V+\langle U\rangle(0;\mathbf{r})\sin 2\phi_V\}\\ e^{D_{VV}(\mathbf{r})/2-2C_{VV}(0)}\{-\langle Q\rangle(0;\mathbf{r})\sin 2\phi_V+\langle U\rangle(0;\mathbf{r})\cos 2\phi_V\}\\ e^{-D_{VV}(\mathbf{r})/2}[-\langle I\rangle(0;\mathbf{r})\sinh D_{IV}(\mathbf{r})+\langle V\rangle(0;\mathbf{r})\cosh D_{IV}(\mathbf{r})]\end{pmatrix}. \tag{3.16}$$

The mean values of the Stokes parameters I and V are equal to their incident values; this may be seen by setting $\mathbf{r}$ equal to zero in Eq. (3.15), and noting $D_{\sigma\sigma'}(0)=0$ according to Eq. (3.9). However, even if the initial circular polarization is zero, its visibility, $\langle V\rangle(z;\mathbf{r})$ is nonetheless nonzero by virtue of the difference between $D^2{}_{+}(\mathbf{r})$ and $D^2{}_{-}(\mathbf{r})$ at nonzero $\mathbf{r}$. An interferometer may, as opposed to a single dish, therefore detect circular polarization from a source even if its radiation is intrinsically unpolarized. This effect was identified by Kukushkin and Ol'yak [27,28]. The depolarization of the linear Stokes visibilities due to stochastic Faraday rotation is manifested through the term $\exp[-2C_{VV}(0)]$ [9].

In a simple model for rotation measure fluctuations in a homogeneous magnetic field (see Sec. IV C), the resulting circularly polarized visibility $\langle V\rangle(z;\mathbf{r})$ is of order α times smaller than the mean intensity for initially unpolarized radiation. Effects of order α are far too small to be of interest for the ISM. Nevertheless, it is of formal interest to interpret their origin. In a medium with a homogeneous magnetic field one has $\delta\phi_V=\alpha\phi_I$, and the phase structure function for the right-hand polarized wave front is $(1+\alpha)^2D_{II}(\mathbf{r})$ and that for the left-hand polarized wave front is $(1-\alpha)^2D_{II}(\mathbf{r})$. Thus the scale length over which each wave front experiences a root-mean-square phase difference of 1 rad differs slightly. This is interpreted in terms of each sense of circular polarization corresponding to different parameters, $r_{\mathrm{diff}+}$ and $r_{\mathrm{diff}-}$, say, for the diffractive scales. If α is positive, one has $r_{\mathrm{diff}+}<r_{\mathrm{diff}-}$, and the left-hand polarized visibilities extend to larger baselines than the right-hand polarized visibilities.

IV. FOURTH-ORDER MOMENTS OF THE WAVE FIELD

The discussion in Sec. III refers to the ensemble averages of the Stokes visibilities. In this section we derive the variance of the visibilities, and hence of the Stokes parameters themselves. The underlying idea is that $\langle V^2\rangle\neq 0$ and $\langle V\rangle=0$ can lead to an observable circular polarization, provided that the time scale for the fluctuations in V is long compared with the time scale for an observation.

A. Solution for statistically homogeneous fluctuations

The fourth-order moment of the wave field describes the correlations of the Stokes visibilities. Using the definitions of the Stokes parameters in Eq. (3.2), the autocorrelation and cross-correlation of the Stokes visibilities may be expressed in terms of the following generalized fourth-order moment of the electric field:

$$\gamma_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(z;\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_1',\mathbf{r}_2')=\langle u_{\sigma_1}(z;\mathbf{r}_1)u_{\sigma_2}(z;\mathbf{r}_2)u^*_{\sigma_1'}(z;\mathbf{r}_1')u^*_{\sigma_2'}(z;\mathbf{r}_2')\rangle. \tag{4.1}$$

This moment describes the cross-correlation in the electric field between four receivers at positions $\mathbf{r}_1$, $\mathbf{r}_1'$, $\mathbf{r}_2$, and $\mathbf{r}_2'$, each receiver measuring either the right- or left-hand circularly polarized component of the radiation according to the sign of the subscript $\sigma_1,\sigma_2,\sigma_1',\sigma_2'=\pm 1$.

Equations (2.12) are used to derive the following equation for generalized fourth-order moment (actually 16 equations for the 16 moments):

$$\left(2ik\frac{\partial}{\partial z}+\nabla_1^2+\nabla_2^2-\nabla_1'^2-\nabla_2'^2 + ikG'_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(z;\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_1',\mathbf{r}_2')\right) \times \gamma_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(z;\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_1',\mathbf{r}_2')=0, \tag{4.2}$$

with $G'_{\sigma_1\sigma_2\sigma_1'\sigma_2'}$ being the ensemble average over the phase fluctuations. For $\sigma_1+\sigma_2-\sigma_1'-\sigma_2'\neq 0$, the following result holds:

$$\begin{aligned} G'_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(z;\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_1',\mathbf{r}_2') &= D'_{z;\sigma_1\sigma_1'}(\mathbf{r}_1-\mathbf{r}_1')+D'_{\sigma_1\sigma_2'}(z;\mathbf{r}_1-\mathbf{r}_2')+D'_{\sigma_2\sigma_1'}(z;\mathbf{r}_2-\mathbf{r}_1') \\ &\quad +D'_{\sigma_2\sigma_2'}(z;\mathbf{r}_2-\mathbf{r}_2')-D'_{\sigma_1\sigma_2}(z;\mathbf{r}_1-\mathbf{r}_2) \\ &\quad -D'_{z;\sigma_1'\sigma_2'}(\mathbf{r}_1'-\mathbf{r}_2')+2i\phi_V(\sigma_1+\sigma_2-\sigma_1'-\sigma_2'), \end{aligned} \tag{4.3}$$

where the primes on $G_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(z;\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_1',\mathbf{r}_2')$, $D(z;\mathbf{r})$, and ϕ_V denote derivatives with respect to z, and the dependence of ϕ_V' on z is implicit.

Changing coordinates to

$$\begin{aligned} \mathbf{R}&=\frac{1}{4}(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_1'+\mathbf{r}_2'), \\ \mathbf{r}&=\mathbf{r}_1+\mathbf{r}_2-\mathbf{r}_1'-\mathbf{r}_2', \\ \boldsymbol{\rho}_1&=\frac{1}{2}(\mathbf{r}_1-\mathbf{r}_2+\mathbf{r}_1'-\mathbf{r}_2'), \\ \boldsymbol{\rho}_2&=\frac{1}{2}(\mathbf{r}_1-\mathbf{r}_2-\mathbf{r}_1'+\mathbf{r}_2'), \end{aligned} \tag{4.4}$$

it is evident that $\Gamma_{\sigma_1\sigma_2\sigma_1'\sigma_2'}$ does not depend on $\mathbf{R}$ in a homogeneous medium, and this also enables us to eliminate $\mathbf{r}$ as a parameter. In view of this simplification it is convenient to change notation, replacing $\gamma_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(z;\mathbf{R},\mathbf{r},\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$ by $\Gamma_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(z;\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$. Equation (4.2) becomes

$$\left[2ik\frac{\partial}{\partial z}+2\nabla_{\rho_1}\cdot\nabla_{\rho_2}+ikG'_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(z;\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)\right] \times\Gamma_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(z;\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)=0, \tag{4.5}$$

$$\begin{aligned} G'_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(z;\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) &= D'_{\sigma_1\sigma_1'}(z;\boldsymbol{\rho}_2)+D'_{\sigma_1\sigma_2'}(z;\rho_1)+D'_{\sigma_2\sigma_1'}(z;\boldsymbol{\rho}_1) \\ &\quad +D'_{\sigma_2\sigma_2'}(z;\boldsymbol{\rho}_2)-D'_{\sigma_1\sigma_2}(z;\boldsymbol{\rho}_1+\boldsymbol{\rho}_2) \\ &\quad -D'_{\sigma_1'\sigma_2'}(z;\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)+2i\phi_V'(\sigma_1+\sigma_2-\sigma_1'-\sigma_2'). \end{aligned} \tag{4.6}$$

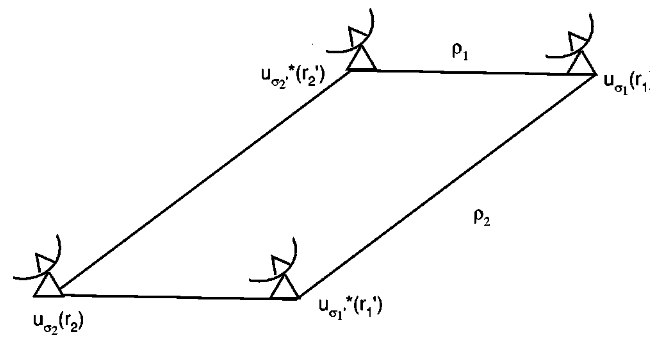

FIG. 1. Positions of the receivers in calculating fourth-order moments [cf. Ishimaru (1978)].

The generalized fourth-order moment $\Gamma_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(z;\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$ describes the correlation between four receivers arranged in a parallelogram whose axes are $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ (see Fig. 1). In an isotropic medium the coordinates $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are interchangeable, $\Gamma_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(z;\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)=\Gamma_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(z;\boldsymbol{\rho}_2,\boldsymbol{\rho}_1)$, (Sec. 20–13 of Ref. [30]), but this is not the case in general in an anisotropic medium.

It is useful to relate the generalized fourth-order moments to autocorrelation and cross-correlation of the Stokes visibilities. The 16 terms in $\Gamma_{\sigma_1\sigma_2\sigma_1'\sigma_2'}$ are separated into four ($\sigma_1=\sigma_1'=\sigma$, $\sigma_2=\sigma_2'=\sigma'$ with $\sigma=\pm 1$, $\sigma'=\pm 1$) that involve only I and V:

$$\Gamma_{\sigma\sigma'\sigma\sigma'}(z,\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)=\langle I^2\rangle(z,\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)+(\sigma+\sigma')\langle IV\rangle(z,\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) +\sigma\sigma'\langle V^2\rangle(z,\boldsymbol{\rho}_1,\boldsymbol{\rho}_2), \tag{4.7}$$

and four ($\sigma_1=\sigma_2=\sigma$, $\sigma_1'=\sigma_2'=\sigma'$ and $\sigma_1=\sigma_2=\sigma$, $\sigma_1'=\sigma_2'=\sigma'$ with $\sigma\neq\sigma'$) that involve only Q and U:

$$\begin{aligned} \Gamma_{\sigma\sigma\sigma'\sigma'}(z,\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)&=\langle Q^2\rangle(z,\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)-2i\sigma\langle QU\rangle(z,\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) -\langle U^2\rangle(z,\boldsymbol{\rho}_1,\boldsymbol{\rho}_2), \\ \Gamma_{\sigma\sigma'\sigma'\sigma}(z,\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)&=\langle Q^2\rangle(z,\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)+\langle U^2\rangle(z,\boldsymbol{\rho}_1,\boldsymbol{\rho}_2). \end{aligned} \tag{4.8}$$

The other eight terms involving cross correlations between I, V and Q, U are not discussed here.

B. Solution for a thin screen

A standard approximation in scintillation theory is to assume that the phase fluctuations in the medium occur on a thin screen located a distance z from the observer [30,31]. We assume that the incident wave front is planar, corresponding to a point source at $z=-\infty$, so that $\langle XY\rangle(0,\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$ is independent of the transverse spatial coordinates $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$. This implies that the fourth-order moments incident on the screen are not functions of $\boldsymbol{\rho}_1$ or $\boldsymbol{\rho}_2$, and we henceforth write $\langle XY\rangle(0)$ for the incident value.

If the source is located at a finite distance it is possible to correct for the spherical nature of the wave front by making the substitutions [31]

$$z\rightarrow\frac{z_1z_2}{z_1+z_2}, \tag{4.9}$$

$$\mathbf{r} \rightarrow \frac{z_1}{z_1+z_2}\mathbf{r}, \tag{4.10}$$

where z_1 is the distance from the source to the scattering screen and z_2 the distance from the screen to the observer. In particular, Eq. (4.10) implies that the length scale of fluctuations on the observer's screen is larger by a factor $(z_1+z_2)/z_1$ compared to the planar case.

The solution of Eq. (4.5) [30] for a planar wave front incident upon a thin screen located a distance z from the observer is

$$\Gamma_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(z,\mathbf{r}_1,\mathbf{r}_2) = \left(\frac{k}{2\pi z}\right)^2 \int d^2\mathbf{r}_1' d^2\mathbf{r}_2' \Gamma_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(0) \times \exp\left[\frac{ik(\mathbf{r}_1-\mathbf{r}_1')\cdot(\mathbf{r}_2-\mathbf{r}_2')}{z} - \frac{1}{2}\int_0^z dz'\, G'_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(z;\mathbf{r}_1',\mathbf{r}_2')\right]. \tag{4.11}$$

With this solution, we use Eqs. (4.7) and (4.8) to assemble solutions for the correlations of the Stokes visibilities as

$$\langle XY\rangle(z,\mathbf{r}_1,\mathbf{r}_2) = \left(\frac{k}{2\pi z}\right)^2 \int d^2\mathbf{r}_1' d^2\mathbf{r}_2' \times \exp\left[\frac{ik(\mathbf{r}_1-\mathbf{r}_1')\cdot(\mathbf{r}_2-\mathbf{r}_2')}{z} - G_{II}(\mathbf{r}_1',\mathbf{r}_2')\right] \times A_{XY}(\mathbf{r}_1',\mathbf{r}_2'), \tag{4.12}$$

$$\begin{pmatrix} A_{II} \\ A_{IV} \\ A_{VV} \end{pmatrix} = \begin{pmatrix} W_{II} & 2W_{IV} & W_{VV} \\ W_{IV} & W_{II}-W_{VV} & W_{IV} \\ W_{VV} & 2W_{IV} & W_{II} \end{pmatrix} \begin{pmatrix} \langle I^2\rangle(0) \\ \langle IV^2\rangle(0) \\ \langle V^2\rangle(0) \end{pmatrix}, \tag{4.13}$$

$$\begin{pmatrix} A_{QQ} \\ A_{QU} \\ A_{UU} \end{pmatrix} = \begin{pmatrix} W_{QQ} & -2W_{QU} & W_{UU} \\ W_{QU} & W_{QQ}-W_{UU} & -W_{QU} \\ W_{UU} & 2W_{QU} & W_{QQ} \end{pmatrix} \begin{pmatrix} \langle Q^2\rangle(0) \\ \langle QU^2\rangle(0) \\ \langle U^2\rangle(0) \end{pmatrix},$$

where the arguments $(\mathbf{r}_1',\mathbf{r}_2')$ are omitted, and we introduce

$$\begin{pmatrix} W_{II} \\ W_{IV} \\ W_{VV} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{-a}\cosh b + e^{a-c} \\ -e^{-a}\sinh b \\ e^{-a}\cosh b - e^{a-c} \end{pmatrix},$$

$$\begin{pmatrix} W_{QQ} \\ W_{QU} \\ W_{UU} \end{pmatrix} = \frac{e^a}{2}\begin{pmatrix} e^{-c'} + d(f+f') \\ id(f-f')/2 \\ e^{-c'} - d(f+f') \end{pmatrix},$$

$$a = G_{VV}(\mathbf{r}_1',\mathbf{r}_2'), \quad b = 2G_{IV}(\mathbf{r}_1',\mathbf{r}_2'),$$

$$c = 2D_{VV}(\mathbf{r}_2'), \quad c' = 2D_{VV}(r_1'),$$

$$d = D_{VV}(\mathbf{r}_1'+\mathbf{r}_2') + D_{VV}(\mathbf{r}_1'-\mathbf{r}_2') - 8C_{VV}(0), \tag{4.14}$$

$$f = \exp[D_{IV}(\mathbf{r}_1'+\mathbf{r}_2') - D_{IV}(\mathbf{r}_1'-\mathbf{r}_2')]e^{-4i\phi_V}/2,$$

$$f' = \exp[-D_{IV}(\mathbf{r}_1'+\mathbf{r}_2') + D_{IV}(\mathbf{r}_1'-\mathbf{r}_2')]e^{4i\phi_V}/2,$$

$$G_{XY}(\mathbf{r}_1,\mathbf{r}_2) = \frac{1}{2}[2D_{XY}(\mathbf{r}_1) + 2D_{XY}(\mathbf{r}_2) - D_{XY}(\mathbf{r}_1+\mathbf{r}_2) - D_{XY}(\mathbf{r}_1-\mathbf{r}_2)],$$

$$X,Y = (I,V). \tag{4.15}$$

In the absence of a magnetic field, an obvious, although important, point is that all the Stokes parameters scintillate like the total intensity. Referring back to the definitions of D_{IV} and D_{VV} in Sec. III, the absence of Faraday rotation terms implies that D_{IV}, D_{VV}, and ϕ_V are zero, leaving only W_{II}, W_{QQ}, and W_{UU} nonzero. In particular, one has $W_{II}=1$ and $W_{QQ}+W_{UU}=1$, so the correlation functions differ only by a multiplicative constant ($\langle XY\rangle(0)$):

$$\langle XY\rangle(z,\mathbf{r}_1,\mathbf{r}_2) = \langle XY\rangle(0)\left(\frac{k}{2\pi z}\right)^2 \int d^2\mathbf{r}_1' d^2\mathbf{r}_2' \times \exp\left[\frac{ik(\mathbf{r}_1-\mathbf{r}_1')\cdot(\mathbf{r}_2-\mathbf{r}_2')}{z} - G_{II}(\mathbf{r}_1',\mathbf{r}_2')\right], \tag{4.16}$$

$$XY = [II,IV,VV,QQ,UU,QU].$$

C. A simple model for rotation measure fluctuations

A simple model is when the fluctuations occur only in the electron density, with the magnetic field being uniform. In this case the structure functions D_{VV} and D_{IV} are related to D_{II} by the parameter $\alpha=\Omega_e/\omega$, where Ω_e is the electron cyclotron frequency and ω is the angular frequency of the radiation. In particular, one has $D_{VV}/\alpha^2 = D_{IV}/\alpha = D_{II}$. Taking a typical value of the magnetic field in the ISM of 3 μG, and an observing frequency of $\nu=1$ GHz, α is of order 10^{-8}.

In this case we expand Eqs. (4.14) in terms of α, where D_{IV}/D_{II} is of order α and D_{VV}/D_{II} is of order α^2. Retaining terms to second order in α, the second-order correlations are given by Eqs. (4.12), with

$$\begin{pmatrix} W_{II} \\ W_{IV} \\ W_{VV} \end{pmatrix} = \begin{pmatrix} 1+b^2/4-c/2 \\ b/2 \\ b^2/4-a+c/2 \end{pmatrix}, \tag{4.17}$$

$$\begin{pmatrix} W_{QQ} \\ W_{QU} \\ W_{UU} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+a-c'+ge^{-8C_{VV}(0)} \\ he^{-8C_{VV}(0)} \\ 1+a-c'-ge^{-8C_{VV}(0)} \end{pmatrix},$$

$$g = \cos 4\phi_V + i\sin 4\phi_V[D_{IV}(\mathbf{r}_1'-\mathbf{r}_2') - D_{IV}(\mathbf{r}_1'+\mathbf{r}_2')] + \frac{1}{2}\cos 4\phi_V\{[D_{IV}(\mathbf{r}_1'-\mathbf{r}_2') - D_{IV}(\mathbf{r}_1'+\mathbf{r}_2')]^2 - D_{VV}(\mathbf{r}_1'+\mathbf{r}_2') - D_{VV}(\mathbf{r}_1'-\mathbf{r}_2') - G_{VV}(\mathbf{r}_1',\mathbf{r}_2')\}, \tag{4.18}$$

$$h=\sin 4\phi_V - i\cos 4\phi_V[D_{IV}(\mathbf{r}_1'-\mathbf{r}_2')-D_{IV}(\mathbf{r}_1'+\mathbf{r}_2')] + \frac{\sin 4\phi_V}{2}\{[D_{IV}(\mathbf{r}_1'-\mathbf{r}_2')-D_{IV}(\mathbf{r}_1'+\mathbf{r}_2')]^2 + D_{VV}(\mathbf{r}_1'+\mathbf{r}_2')+D_{VV}(\mathbf{r}_1'-\mathbf{r}_2')+G_{VV}(\mathbf{r}_1',\mathbf{r}_2')\}.$$

Equations (4.17) and (4.18) are used in the discussion below.

V. IDENTIFICATION OF EFFECTS

In this section we discuss the interpretation of Eqs. (4.12) and (4.17), concentrating on the fluctuations in circular polarization and the total intensity.

Circular polarization

We are concerned with the creation of a circularly polarized component, and so we assume the source has no intrinsic circular polarization, which corresponds to assuming $\langle V^2\rangle(0)=0$ and $\langle IV\rangle(0)=0$ in Eq. (4.12). The variances in the visibility and in the Stokes V visibility are then

$$\langle I^2\rangle(z,\mathbf{r}_1,\mathbf{r}_2)=\langle I^2\rangle(0)\left(\frac{k}{2\pi z}\right)^2\int d^2\mathbf{r}_1'\mathbf{r}_2' \times\exp\left[\frac{ik(\mathbf{r}_1-\mathbf{r}_1')\cdot(\mathbf{r}_2-\mathbf{r}_2')}{z}-G_{II}(\mathbf{r}_1',\mathbf{r}_2')\right] \times\{1+|G_{IV}(\mathbf{r}_1',\mathbf{r}_2')|^2-D_{VV}(\mathbf{r}_2')\}, \tag{5.1}$$

$$\langle V^2\rangle(z,\mathbf{r}_1,\mathbf{r}_2)=\langle I^2\rangle(0)\left(\frac{k}{2\pi z}\right)^2\int d^2\mathbf{r}_1'\mathbf{r}_2' \times\exp\left[\frac{ik(\mathbf{r}_1-\mathbf{r}_1')\cdot(\mathbf{r}_2-\mathbf{r}_2')}{z}-G_{II}(\mathbf{r}_1',\mathbf{r}_2')\right] \times\{-G_{VV}(\mathbf{r}_1,\mathbf{r}_2)+D_{VV}(\mathbf{r}_2') +|G_{IV}(\mathbf{r}_1',\mathbf{r}_2')|^2\}. \tag{5.2}$$

For convenience in the discussion below, we label the contributions to $\langle V^2\rangle$ due to G_{VV}, D_{VV}, and $|G_{IV}|^2$ as $\langle V^2\rangle_1$, $\langle V^2\rangle_2$, and $\langle V^2\rangle_3$ in Eq. (5.2), respectively.

1. Variations in the total intensity

Before discussing the interpretation of Eq. (5.2), we review the subject of intensity variations in an isotropic medium, because many of the approximations and definitions are relevant to the anisotropic case. In the strong scattering regime, intensity variations are due to diffractive scintillation, caused by interference between subimages over the scattering disk, and refractive scintillation, due to refractive focusing and defocusing of the entire scattering disk [6].

Intensity variations in an isotropic medium correspond to the first term in the curly brackets in Eq. (5.1). Following Ref. [32] this term may be written in the form

$$\langle I^2\rangle_{II}(z;\mathbf{r}_1,\mathbf{r}_2)=\left(\frac{k}{2\pi z}\right)^2\int d^2\mathbf{r}_1'd^2\mathbf{r}_2'\langle I^2\rangle(0) \times\exp\left[\frac{ik(\mathbf{r}_1-\mathbf{r}_1')\cdot(\mathbf{r}_2-\mathbf{r}_2')}{z}\right] \times\{e^{-D(\mathbf{r}_1')}[1+\Omega(r_2',r_1')+\cdots] +e^{-D(\mathbf{r}_2')}[1+\Omega(\mathbf{r}_1',\mathbf{r}_2')+\cdots]\}, \tag{5.3}$$

$$\Omega(\mathbf{r}_1,\mathbf{r}_2)=D_{II}(\mathbf{r}_1+\mathbf{r}_2)/2+D_{II}(\mathbf{r}_1-\mathbf{r}_2)/2-D_{II}(\mathbf{r}_1), \tag{5.4}$$

where the subscript II signifies that only isotropic phase fluctuations are retained. The approximation made in deriving Eq. (5.3) follows by recognizing that, for strong scattering, the intensity fluctuations are dominated by regions where either $\mathbf{r}_1'$ or $\mathbf{r}_2'$ are small, and then assuming $r_2 \gg r_1$ or $r_1 \gg r_2$ for these regions respectively. Equation (5.3) is written compactly in the form

$$\langle I^2\rangle_{II}(z,\mathbf{r}_1,\mathbf{r}_2)=[2+2\xi(z,\mathbf{r}_1,\mathbf{r}_2)]\langle I^2\rangle(0). \tag{5.5}$$

The term ξ is due to the contribution of Ω, and is associated with large scale focusing and defocusing of the entire scattering disk, giving rise to so-called ''refractive scintillation'' [6]. The explicit expression for ξ is

$$\xi(z;\mathbf{r}_1,\mathbf{r}_2)=2\int\frac{d^2\mathbf{q}}{(2\pi)^2}\Phi(\mathbf{q})\{1-\cos(\mathbf{q}\cdot\boldsymbol{\rho}_1-q^2r_F^2)\} \times\exp[i\mathbf{q}\cdot\boldsymbol{\rho}_2-D_{II}(\boldsymbol{\rho}_1-\mathbf{q}r_F^2)]. \tag{5.6}$$

This term is less than unity in the strong scattering régime [6]. The power spectrum of phase inhomogeneities, $\Phi(\mathbf{q})$, is defined by

$$D_{II}(r)=2\int\frac{d^2\mathbf{q}}{(2\pi)^2}\Phi(\mathbf{q})[1-e^{i\mathbf{q}\cdot\mathbf{r}}]. \tag{5.7}$$

A power law spectrum $\Phi(\mathbf{q})$ has the form

$$\Phi(\mathbf{q})=Q_{II}q^{-\beta},\quad q_{\min}<q<q_{\max}, \tag{5.8}$$

where $q_{\min}$ and $q_{\max}$ correspond to the inverses of the outer and inner scales of the scattering medium, $r_{\rm out}$ and $r_{\rm in}$, respectively, and Q_{II} is a constant. The familiar case of Kolmogorov turbulence corresponds to $\beta=\frac{11}{3}$. Relating the definition of the phase structure function $D_{II}(\mathbf{r})$ provided by Ref. [33] to the power spectrum of phase inhomogeneities [34], one has

$$Q_{II}=8\pi^3\Delta L C_N^2 r_e^2\lambda^2, \tag{5.9}$$

where r_e is the classical radius of the electron, ΔL is the path length through the scattering medium, and C_N^2 is the electron density structure constant. Form (5.8) for $\Phi(\mathbf{q})$ is used extensively below.

Inspection of Eq. (5.5) reveals that the variance of the intensity scintillations,

$$\langle I^2\rangle(z,0,0)-\langle I^2\rangle(0)=[1+2\xi(z,0,0)]\langle I^2\rangle(0), \tag{5.10}$$

contains contributions from two terms. The term $2\xi(z,0,0)\langle I^2\rangle(0)$ is associated with refractive scintillation, while the term of magnitude $\langle I^2\rangle(0)$ is identified with diffractive scintillation.

2. Circular polarization correction terms

We now consider the three terms in Eq. (5.2), assuming that fluctuations in the rotation measure are determined by fluctuations in the density alone, implying $D_{II}=D_{IV}/\alpha=D_{VV}/\alpha^2$, with $\alpha=\Omega_e/\omega$.

The contribution from the first term, denoted $\langle V^2\rangle_1$, comes from the regions where either $\mathbf{r}_1'$ or $\mathbf{r}_2'$ are small. Since $G_{VV}=\alpha^2 G_{II}$, this term may be approximated by referring to Eqs. (5.3) and (5.6),

$$\langle V^2\rangle_1(z,\mathbf{r}_1,\mathbf{r}_2)=\alpha^2\left(\frac{K}{2\pi z}\right)^2\int d\mathbf{r}_1' d\mathbf{r}_2'\langle I^2\rangle(0)\times\exp\left[\frac{ik(\mathbf{r}_1-\mathbf{r}_1')\cdot(\mathbf{r}_2-\mathbf{r}_2')}{z}\right]\times\{\exp[-D_{II}(r_1')]\Omega(\mathbf{r}_2',\mathbf{r}_1')+\exp[-D_{II}(\mathbf{r}_2')]\Omega(\mathbf{r}_1',\mathbf{r}_2')\},\tag{5.11}$$

which may be rewritten as

$$\langle V^2\rangle_1(z,\mathbf{r}_1,\mathbf{r}_2)=\alpha^2\langle I^2\rangle(0)[\xi(z,\mathbf{r}_1,\mathbf{r}_2)+\xi(z,\mathbf{r}_2,\mathbf{r}_1)],\tag{5.12}$$

with ξ given by Eq. (5.6). Production of circular polarization due to the term $\langle V^2\rangle_1$ is associated with the correction term in Eq. (5.6) due to refractive intensity variations.

It is possible to derive an expression for $\langle V^2\rangle_1$ from Eq. (5.11) directly. Consider the first term in curly brackets in Eq. (5.11). The exponential term is only large for small $\mathbf{r}_1'$, so it is appropriate to expand $\Omega(\mathbf{r}_2',\mathbf{r}_1')$ for small r_1/r_2:

$$\Omega(\mathbf{r}_2',\mathbf{r}_1')\approx\left(\frac{r_2'}{r_{\rm diff}}\right)^{\beta-2}(\beta-3)(\beta-2)\left(\frac{r_1'}{r_2'}\right)^2.\tag{5.13}$$

Specializing to the case $\mathbf{r}_1=\mathbf{r}_2=0$, the terms involving $\mathbf{r}_1'$ and $\mathbf{r}_2'$ in curly brackets in Eq. (A3) are interchangeable. One then has

$$\langle V^2\rangle_1(z,0,0)=2\alpha^2\langle I^2\rangle(0)\left(\frac{k}{2\pi z}\right)^2\int d\mathbf{r}_1'(\beta-3)(\beta-2)r_1^2\times\left(\frac{1}{r_{\rm diff}}\right)^{\beta-2}\int dr_2'\exp\left[\frac{ik\mathbf{r}_1'\cdot\mathbf{r}_2'}{z}\right]\mathbf{r}_2'^{\beta-4}.\tag{5.14}$$

We evaluate the integral over $\mathbf{r}_2'$ using [35]

$$\int d^2\mathbf{r}_2'\exp[i\mathbf{x}\cdot\mathbf{r}_2']r_2^{\gamma}=\pi 2^{\gamma+2}\frac{\Gamma(\gamma/2+1)}{\Gamma(-\gamma/2)}x^{-\gamma-2},\tag{5.15}$$

which is valid for $\gamma>-2$. The remaining integral over $\mathbf{r}_1'$ yields

$$\langle V^2\rangle_1(z,0,0)=\alpha^2\langle I^2\rangle(0)2^{\beta-3}\pi^{-1}(\beta-3)(\beta-2)\times\left(\frac{r_F}{r_{\rm diff}}\right)^{2\beta-8}\frac{\Gamma(\beta/2-1)}{\Gamma(2-\beta/2)}\Gamma\left(\frac{6-\beta}{\beta-2}\right).\tag{5.16}$$

Since the modulation index due to refractive intensity variations, $m_{\rm ref}$, is of order $(r_F/r_{\rm diff})^{4-\beta}$ (see, e.g., Ref. [31]), Eq. (5.16) implies a degree of circular polarization of order $\alpha m_{\rm ref}\langle I^2\rangle^{1/2}$. The Appendix details the contribution of $\langle V^2\rangle_1$ when the rotation measure fluctuations are not necessarily proportional to the isotropic phase fluctuations.

The contribution due to the term $\langle V^2\rangle_2$ is also evaluated by representing the rotation measure structure function in terms of its power spectrum. Evaluating the integrals over $\mathbf{r}_1'$ and $\mathbf{r}_2'$ in Eq. (5.2), the variance in circular polarization due to the term $D_{VV}(\mathbf{r}_2')$ is

$$\langle V^2\rangle_2(z,\mathbf{r}_1,\mathbf{r}_2)=2\alpha^2\langle I^2\rangle(0)\int d^2\mathbf{q}\frac{\Phi(\mathbf{q})}{(2\pi)^2}\times\left[\exp[-D(r_1)]-\frac{1}{2}\{e^{i\mathbf{q}\cdot\mathbf{r}_2}\times\exp[-D(\mathbf{r}_1-r_F^2\mathbf{q})+e^{-i\mathbf{q}\cdot\mathbf{r}_2}\times\exp[-D(\mathbf{r}_1+r_F^2\mathbf{q})\}\right].\tag{5.17}$$

For arbitrary $\mathbf{r}_1$ and $\mathbf{r}_2$ this expression is, in general, complex, and unlike $\langle V^2\rangle_1(z,\mathbf{r}_1,\mathbf{r}_2)$, it is not symmetric under interchange of $\mathbf{r}_1$ and $\mathbf{r}_2$. The circular polarization is a real quantity, and Eq. (5.17) can be directly expressed in terms of the Stokes parameter V only for $\mathbf{r}_2=0$, in which case the expression is real, as required.

The contribution from $\langle V^2\rangle_2$ to the variance in circular polarization is obtained by setting $\mathbf{r}_1=\mathbf{r}_2=0$ in Eq. (5.17). The integral is approximated using

$$[1-\exp[-D(\mathbf{q}r_F^2)]]\approx\begin{cases}D(\mathbf{q}r_F^2), & q_{\rm min}<q<1/r_{\rm ref}\\ 1, & 1/r_{\rm ref}<q<q_{\rm max},\end{cases}\tag{5.18}$$

to yield

$$\langle V^2\rangle_2(z,0,0)=\frac{\alpha^2\langle I^2\rangle(0)}{\pi}(r_{\rm ref})^{\beta-2}\left[\frac{1}{\beta-2}+\ln\left(\frac{1}{q_{\rm min}r_{\rm ref}}\right)\right],\tag{5.19}$$

where we assume that the outer scale is much larger than the refractive scale ($L_0\gg r_{\rm ref}$). Rewriting Q_{II} in terms of $r_{\rm diff}$, one finds the magnitude of the circular polarization,

$$\langle V^2\rangle_2(z,0,0)=\alpha^2\langle I^2\rangle(0)f(\beta)\left(\frac{r_F}{r_{\rm diff}}\right)^{2\beta-4}\times\left[\frac{1}{\beta-2}+\ln\left(\frac{1}{q_{\rm min}r_{\rm ref}}\right)\right],\tag{5.20}$$

with

$$f(\beta)=-2^{\beta-1}\frac{\Gamma(\beta/2)}{\Gamma(1-\beta/2)}, \tag{5.21}$$

for $\beta<4$. For $\beta=\frac{11}{3}$ one has $f(\beta)\approx 0.89$. The contribution of the circular polarization due to this term scales as $\alpha^2(r_{\rm ref}/r_{\rm diff})^{\beta-2}$.

The contribution of $\langle V^2\rangle_3$ in a homogeneous magnetic field is evaluated by appealing to the arguments presented in the evaluation of $\langle V^2\rangle_1$. For strong scattering, only the regions near $\mathbf{r}_1'\approx 0$ and $\mathbf{r}_2'\approx 0$ contribute. Expanding for $r_1'\ll r_2'$ and $r_2'\ll r_1'$, one has

$$\langle V^2\rangle_3(z,\mathbf{r}_1,\mathbf{r}_2)=\alpha^2\langle I^2\rangle(0)\left(\frac{k}{2\pi z}\right)^2\int d^2\mathbf{r}_1'd^2\mathbf{r}_2' \times\exp\left[\frac{ik(\mathbf{r}_1-\mathbf{r}_1')\cdot(\mathbf{r}_2-\mathbf{r}_2')}{z}\right] \times\{e^{-D(r_1')}\Omega^2(\mathbf{r}_2',\mathbf{r}_1')+e^{-D(\mathbf{r}_2')}\Omega^2(\mathbf{r}_1',\mathbf{r}_2')\}. \tag{5.22}$$

Consider the second term in curly brackets in Eq. (5.22). The exponential ensures that this term is only large for small r_2', so one may expand $\Omega^2(\mathbf{r}_1',\mathbf{r}_2')$, defined by Eq. (5.4), for small r_2'. To second order in r_2'/r_1' one has

$$\Omega^2(\mathbf{r}_1',\mathbf{r}_2')\approx\left(\frac{r_1'}{r_{\rm diff}}\right)^{2(\beta-2)}\frac{(\beta-2)^2(\beta-3)^2r_2'^4}{4r_1'^4}. \tag{5.23}$$

Recognizing that the two terms inside the curly brackets in Eq. (5.22) are identical under the replacement $\mathbf{r}_1'\leftrightarrow\mathbf{r}_2'$, provided $\mathbf{r}_1=\mathbf{r}_2=0$, one then has

$$\langle V^2\rangle_3(z,0,0)\approx 2\left(\frac{k}{2\pi z}\right)^2\alpha^2\int d^2\mathbf{r}_1'd^2\mathbf{r}_2' \times\exp\left[\frac{ik\mathbf{r}_1'\cdot\mathbf{r}_2'}{z}-D_{II}(\mathbf{r}_1)\right] \times\frac{r_1^4D_{II}^2(\mathbf{r}_2)}{r_2^4}\frac{(\beta-2)^2(\beta-3)^2}{4}. \tag{5.24}$$

Now we may evaluate the integral over $\mathbf{r}_2'$ using Eq. (5.15). Then Eq. (5.22) reduces to

$$\langle V^2\rangle_3=\pi\langle I^2\rangle(0)\frac{\alpha^2(\beta-2)^2(\beta-3)^2}{8\pi^2}r_F^{4\beta-16}r_{\rm diff}^{4-2\beta}2^{2\beta-6} \times\frac{\Gamma(\beta-3)}{\Gamma(4-\beta)}\int d^2\mathbf{r}_1r_1^{10-2\beta}\exp[-D_{II}(r_1')]. \tag{5.25}$$

Now, using

$$\int_0^\infty dr_1r_1^\gamma\exp[-(r/r_{\rm diff})^{\beta-2}]=r_{\rm diff}^{1+\gamma}\frac{1}{\beta-2}\Gamma\left(\frac{1+\gamma}{\beta-2}\right), \tag{5.26}$$

which is valid for $\beta>2$, one has

$$\langle V^2\rangle_3=\alpha^2\langle I^2\rangle(0)(\beta-2)(\beta-3)^2 2^{2\beta-7} \times\left(\frac{r_F}{r_{\rm diff}}\right)^{4\beta-16}\Gamma\left(\frac{12-2\beta}{\beta-2}\right)\frac{\Gamma(\beta-3)}{\Gamma(4-\beta)}. \tag{5.27}$$

For strong scattering, the term $(r_F/r_{\rm diff})^{4\beta-16}$ is less than unity. In particular, for a Kolmogorov spectrum of turbulent fluctuations ($\beta=\frac{11}{3}$), the root mean square degree of circular polarization due to this term scales as $\alpha(r_{\rm diff}/r_F)^{2/3}$, less than unity for strong scattering.

The dominant term is $\langle V^2\rangle_2\gg\langle V^2\rangle_1,\langle V^2\rangle_3$ as this is the only term that allows $\sqrt{\langle V^2\rangle}/\langle I\rangle\alpha$ to be much greater than unity. We retain only this dominant term in the discussion below.

3. Cross Correlation

So far we discuss only the variance $\langle V^2\rangle$ in Stokes V. The correlation function $\langle IV\rangle$ contains additional information about the propagation-induced circular polarization.

The form of the expression for $\langle IV\rangle(z,\mathbf{r}_1,\mathbf{r}_2)$ in Eq. (4.12), in the absence of any intrinsic circular polarization, is of the same form as the expression for $\langle V^2\rangle_1$, yielding [cf. Eq. (5.11)]

$$\langle IV\rangle(z,0,0)=\langle I^2\rangle(z,0,0)\left(\frac{k}{2\pi z}\right)^2\int d^2\mathbf{r}_1'd^2\mathbf{r}_2' \times\exp\left[\frac{ik\mathbf{r}_1'\cdot\mathbf{r}_2'}{z}\right]\{\exp[-D(\mathbf{r}_1')]\Omega_{IV}(\mathbf{r}_2',\mathbf{r}_1') +\exp[-D(\mathbf{r}_2')]\Omega_{IV}(\mathbf{r}_1',\mathbf{r}_2')\}, \tag{5.28}$$

$$\Omega_{IV}(\mathbf{r}_1,\mathbf{r}_2)=D_{IV}(\mathbf{r}_1+\mathbf{r}_2)+D_{IV}(\mathbf{r}_1-\mathbf{r}_2)-D_{IV}(\mathbf{r}_1). \tag{5.29}$$

For a homogeneous magnetic field, one has $\Omega_{IV}(\mathbf{r}_1,\mathbf{r}_2)=\alpha\Omega(\mathbf{r}_1,\mathbf{r}_2)$ and the cross-correlation is

$$\langle IV\rangle(z,0,0)=2\alpha\xi(z,0,0)\langle I^2\rangle(0). \tag{5.30}$$

Since $\xi(z,0,0)$ is of order unity, it follows that the cross correlation is of order α, rendering this correlation too small to be observed.

VI. DISCUSSION AND CONCLUSIONS

Scintillations in a magnetized ISM necessarily lead to a small degree of circular polarization (CP) even for an unpolarized source. This occurs because the refractive indices for the two natural wave modes are slightly different, leading to a variety of small effects that are different for the two opposite CP components. Under the assumption that the fluctuations involve only the plasma density, these effects are characterized by a single small parameter, $\alpha=\Omega_e/\omega$, which is the ratio of the electron cyclotron frequency at the scattering screen to the wave frequency. For typical parameters in the ISM, $\alpha\sim 10^{-8}$ is too small for the induced CP to be of practical interest unless the effect can be enhanced in some way.

The effect that we identify as of possible practical interest appears in the variance of the Stokes parameter V, denoted $\langle V^2\rangle$. Provided that the time scale of observation is short compared with the time scale on which $\langle V^2\rangle$ changes, a net circular polarization of order $\langle V^2\rangle^{1/2}/\langle I\rangle$ should be observed. In evaluating $\langle V^2\rangle$ we separate it into three terms, $\langle V^2\rangle_i$, $i=1$, 2, and 3, and find that each of them is of the form $\langle V^2\rangle_i=\alpha^2\langle I^2\rangle(0)A_i(r_F/r_{\rm diff})^{a_i}$, with A_i all of order unity. For strong scattering, $r_F\gg r_{\rm diff}$ implies $\langle V^2\rangle\gg\alpha^2\langle I^2\rangle$ provided that a_i is not too small. For the term $\langle V^2\rangle_2$, with $a_2=2\beta-4=10/3$ for a Kolmogorov spectrum $\beta=11/3$, the numerical factor is large when the scattering is strong. This term arises from scattering due to the largest structures in the postulated power-law spectrum of fluctuations.

An obvious prediction of this theory is that the observed degree of CP should reverse randomly on the time scale over which $\langle V^2\rangle$ changes. However, the simplifying assumptions we make need to be reconsidered in making a realistic estimate of both the magnitude and time scale of the fluctuations in V. In particular, we assume that both the fluctuations that lead to scintillations in the usual sense and the fluctuation in rotation measure that cause a separation in the right and left polarizations occur at a single phase screen. The assumption that these two effects occur together is unnecessarily restrictive. The effect that we describe requires (a) that the wave front be rippled; and (b) that the right and left polarizations be separated, but there is no need for these two effects to be due to the same structures in the ISM. Indeed, the dominant contribution to $\langle V^2\rangle$ is due to the large-scale structures in the assumed turbulence, and in fact any large-scale magnetized structure can act as a ‘‘Faraday wedge’’ in refracting the right and left polarizations into slightly different directions. In principle, the separation between the two circularly polarized rays can become arbitrarily large at arbitrarily large distances from such a Faraday wedge, leading to nonoverlapping right and left-hand polarized images, and so to $\langle V^2\rangle\sim\langle(I-\langle I\rangle)^2\rangle$. We propose to develop this idea further and apply it to the interpretation of observed CP elsewhere.

Furthermore, an exact theory requires knowledge of the turbulent fluctuations in both the orientation and magnitude of the magnetic field (here assumed to be homogeneous), and its correlation with density fluctuations. The turbulence in the ISM is expected to be highly anisotropic with respect to the magnetic field lines. As remarked in Ref. [36], the fact that the VLBI images of scatter-broadened images are typically only slightly elliptical (with axial ratios at most $\sim$2:1) implies that the radiation propagates through many regions with different magnetic field orientations. This is expected to enhance the production of scintillation-induced CP.

We conclude (1) that scintillation-induced CP must occur due to propagation through the magnetized ISM; (2) that the simplest estimate that it is of order $\alpha\sim10^{-8}$, and so would be unobservably small, is correct only for the mean value of $\langle V\rangle/\langle I\rangle$; (3) that the variance $\langle V^2\rangle/\langle(I-\langle I\rangle)^2\rangle$ can be much larger than α^2; and (4) that a more general model that treats the scintillations and the separation of the opposite circularly polarized rays (‘‘Faraday wedge’’) in different ways is needed in formulating a theory that will be a useful basis for comparison with observations.

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APPENDIX: CONTRIBUTION OF $\langle V^2\rangle_1$

We evaluate the contribution of the $\langle V^2\rangle_1$ term when the structure functions D_{II} and D_{VV} are not necessarily proportional to each other. Writing

$$D_{VV}(\mathbf{r})=\left(\frac{T}{r_{{\rm diff}\,R}}\right)^{\beta_V-2}, \tag{A1}$$

the contribution of $\langle V^2\rangle_1$ is

$$\langle V^2\rangle_1(z,\mathbf{r}_1,\mathbf{r}_2)=\int d\mathbf{r}_1'd\mathbf{r}_2'\langle I^2\rangle(0)\times\exp\left[\frac{ik(\mathbf{r}_1-\mathbf{r}_1')\cdot(\mathbf{r}_2-\mathbf{r}_2')}{z}\right] \tag{A2}$$

$$\times\{\exp[-D_{II}(\mathbf{r}_1')]\Omega_{VV}(\mathbf{r}_2',\mathbf{r}_1')+\exp[-D_{II}(\mathbf{r}_2')]\Omega_{VV}(\mathbf{r}_1',\mathbf{r}_2')\}, \tag{A3}$$

where we write

$$\Omega_{VV}(\mathbf{r}_2,\mathbf{r}_1)=D_{VV}(\mathbf{r}_1+\mathbf{r}_2)/2+D_{VV}(\mathbf{r}_1-\mathbf{r}_2)/2-D_{VV}(\mathbf{r}_2). \tag{A4}$$

Consider the first term in curly brackets in Eq. (A3). The exponential term is only large for small $\mathbf{r}_1$, so it is appropriate to expand $\Omega_{VV}(\mathbf{r}_2,\mathbf{r}_1)$ for small r_1/r_2:

$$\Omega_{VV}(\mathbf{r}_2,\mathbf{r}_1)\approx\left(\frac{r_2}{r_{{\rm diff}\,R}}\right)^{\beta_V-2}(\beta_V-3)(\beta_V-2)\left(\frac{r_1}{r_2}\right)^2. \tag{A5}$$

Specializing to the case $\mathbf{r}_1=\mathbf{r}_2=0$, the terms involving $\mathbf{r}_1'$ and $\mathbf{r}_2'$ in curly brackets in Eq. (A3) are interchangeable. One then has

$$\langle V^2\rangle_1(z,0,0)=2\langle I^2\rangle(0)\left(\frac{k}{2\pi z}\right)^2\int d\mathbf{r}_1'(\beta_V-3)(\beta_V-2)r_1^2\times\left(\frac{1}{r_{{\rm diff}\,R}}\right)^{\beta_V-2}\int d\mathbf{r}_2'\exp\left[\frac{ik\mathbf{r}_1'\cdot\mathbf{r}_2'}{z}\right]\mathbf{r}_2'^{\,\beta_V-4}. \tag{A6}$$

We evaluate the integral over $\mathbf{r}_2'$ using Eq. (5.15). The remaining integral over $\mathbf{r}_1'$ is evaluated to yield

$$\langle V^2\rangle_1(z,0,0)=\langle I^2\rangle(0)2^{\beta_V-3}\pi^{-1}(\beta_V-3)(\beta_V-2)r_F^{2\beta_V-8}\times r_{{\rm diff}\,R}^{2-\beta_V}r_{\rm diff}^{6-\beta_V}\frac{\Gamma(\beta_V/2-1)}{\Gamma(2-\beta_V/2)}\Gamma\left(\frac{6-\beta_V}{\beta-2}\right). \tag{A7}$$

Setting $D_{VV}(\mathbf{r})=\alpha^2D_{II}(\mathbf{r})$, one then has $\beta_V=\beta$ and $r_{{\rm diff}\,R}^{2-\beta}=\alpha^2r_{\rm diff}^{2-\beta}$, and Eq. (A7) is then proportional to the variance due to refractive scintillation, of order $(r_F/r_{\rm diff})^{8-2\beta}$, which is consistent with Eq. (5.16).

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